

Finite $SU(N)^k$ Unification

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ABSTRACT: We consider $N = 1$ supersymmetric gauge theories based on the group $SU(N)_1 \times SU(N)_2 \times \dots \times SU(N)_k$ with matter content $(N, N^*, 1, \dots, 1) + (1, N, N^*, \dots, 1) + \dots + (N^*, 1, 1, \dots, N)$ as candidates for the unification symmetry of all particles. In particular we examine to which extent such theories can become finite and we find that a necessary condition is that there should be exactly three families. We discuss further some phenomenological issues related to the cases $(N, k) = (3, 3)$, $(3, 4)$, and $(4, 3)$, in an attempt to choose those theories that can become also realistic. Thus we are naturally led to consider the $SU(3)^3$ model which we first promote to an all-loop finite theory and then we study its additional predictions concerning the top quark mass, Higgs mass and supersymmetric spectrum.

KEYWORDS: finiteness, GUT, bsm, suy, qkm.

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Contents

1. Introduction	1
2. Reduction of Couplings and Finiteness in $N = 1$ Supersymmetric Gauge Theories	2
3. Soft Supersymmetry Breaking in $N = 1$ FUTS	3
4. Search for realistic FUTs based on product gauge groups	4
5. An all-loop $SU(3)^3$ FUT	7
6. Predictions and Conclusions	10
7. Acknowledgements	12

1. Introduction

Finite field theories [1, 2, 3, 4, 5] are very attractive since they are free of all ultraviolet divergencies, but require a large degree of symmetry, which obviously is not observed at low energies. However, the intriguing possibility exists that the standard model (SM) we observe is a remnant of a finite Grand Unified Theory (GUT) at the unification scale and above. This may provide the missing deep connection of current phenomenology with string theory and may point to a unique candidate for the description of all fundamental interactions.

Finite Unified Theories (FUTs) are $N = 1$ supersymmetric GUTs, which can be made finite even to all-loop orders, including the soft supersymmetry breaking sector. The method to construct GUTs with reduced independent parameters [6, 7] consists of searching for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. Of particular interest is the possibility to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [1, 2]. In order to achieve the latter it is enough to study the uniqueness of the solutions to the one-loop finiteness conditions [1, 2, 5]. Using the above tools elegant $N = 1$ supersymmetric $SU(5)$ examples already exist, and have predicted correctly from the dimensionless sector, among others, the top quark mass [3, 4]. The search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories [8, 9], which involves parameters of dimension one and two.

Here we examine the construction of realistic FUTs based on product gauge groups. In particular we point out that finiteness actually determines the number of families n_f

in a class of supersymmetric $SU(N)^k$ gauge theories, namely $n_f = 3$ regardless of N and k . The case $N = 4$ and $k = 3$ was first pointed in ref. [10], and that of arbitrary N and $k = 3$ was discussed in ref. [11], both from the string point of view. Concerning the soft supersymmetry breaking sector of these latter models, although in principle it could be understood too in the same framework under certain assumptions [10, 12, 13], the explicit construction is still missing.

Our search for realistic FUTs based on product groups leads us to choose a supersymmetric $SU(3)^3$ model, which we subsequently promote to an all-loop finite theory, whose predictions we examine further.

The rest of the paper is organised as follows. In section 2 we review the method of reduction of couplings and recall how it is applied in $N = 1$ supersymmetric gauge theories in order to obtain all-loop finite gauge theories. In section 3 we describe the extension of finiteness in the case of soft supersymmetry breaking terms. Section 4 is devoted to a search for realistic FUTs based on product groups, out of which an $SU(3)^3$ supersymmetric gauge theory with three families is singled out. This theory then is further discussed in detail in section 5. Section 6 contains the predictions of the $SU(3)^3$ FUT concerning the top quark mass, the Higgs boson masses and the supersymmetric spectrum.

2. Reduction of Couplings and Finiteness in $N = 1$ Supersymmetric Gauge Theories

Let us first recall the basic issues concerning *reduction of couplings*, in the case of dimensionless couplings and *finiteness* of $N = 1$ supersymmetric theories.

A RGI relation among couplings g_i ,

$$\mathcal{F}(g_1, \dots, g_N) = 0, \quad (2.1)$$

has to satisfy the partial differential equation

$$\mu \, d\mathcal{F}/d\mu = \sum_{i=1}^N \beta_i \partial \mathcal{F} / \partial g_i = 0, \quad (2.2)$$

where β_i is the β -function of g_i . There exist $(N - 1)$ independent \mathcal{F} 's, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [6],

$$\beta_g (dg_i/dg) = \beta_i, \quad i = 1, \dots, N, \quad (2.3)$$

where g and β_g are the primary coupling and its β -function. Using all the $(N - 1)$ \mathcal{F} 's to impose RGI relations, one can in principle express all the couplings in terms of a single coupling g . The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

In order to discuss finiteness, it seems unavoidable that we should consider supersymmetric gauge theories. Let us then consider a chiral, anomaly free, $N = 1$ globally

supersymmetric gauge theory based on a group G with gauge coupling constant g . The superpotential of the theory is given by

$$W = \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k, \quad (2.4)$$

where m^{ij} and C^{ijk} are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G . All the one-loop β -functions of the theory vanish if $\beta_g^{(1)}$ and all the anomalous dimensions of the superfields $\gamma_i^{j(1)}$ vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G), \quad \frac{1}{2} C_{ipq} C^{j pq} = 2\delta_i^j g^2 C_2(R_i), \quad (2.5)$$

where $\ell(R_i)$ is the Dynkin index of R_i , and $C_2(G)$, $C_2(R_i)$ are respectively the quadratic Casimir invariant of the adjoint representation of G , and of the R_i representation. A natural question to ask is what happens at higher loop orders. A very interesting result is that the conditions (2.5) are necessary and sufficient for finiteness at the two-loop level [14, 15].

The one- and two-loop finiteness conditions (2.5) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (2.4). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a $U(1)$ gauge group is incompatible with the first of the conditions (2.5), due to $C_2[U(1)] = 0$. This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. Specifically there exists a very interesting theorem [1] which guarantees the vanishing of the β -functions to all orders in perturbation theory, if we demand reduction of couplings, and that all the one-loop anomalous dimensions of the matter field in the completely and uniquely reduced theory vanish identically.

3. Soft Supersymmetry Breaking in $N = 1$ FUTS

The above described method of reducing the dimensionless couplings has been extended [8] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [16] that RGI SSB scalar masses in general Gauge-Yukawa unified models satisfy a universal sum rule at one-loop, which was subsequently extended first up to two-loops [4] and then to all-loops [17].

To be more specific, consider the superpotential given by (2.4) along with the Lagrangian for SSB terms

$$\begin{aligned} -\mathcal{L}_{\text{SB}} = & \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j \\ & + \frac{1}{2} (m^2)_i^j \phi^{*i} \phi_j + \frac{1}{2} M \lambda \lambda + \text{H.c.}, \end{aligned} \quad (3.1)$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since we would like to consider only finite theories here, we assume that the one-loop β -function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form $C^{ijk} = g \sum_{n=0} \rho_{(n)}^{ijk} g^{2n}$. According to the finiteness theorem of ref. [1], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved [14, 18] by imposing the condition

$$h^{ijk} = -M C^{ijk} + \dots = -M \rho_{(0)}^{ijk} g + O(g^5). \quad (3.2)$$

In addition it was found [4] that one and two-loop finiteness requires that the following two-loop sum rule for the soft scalar masses has to be satisfied

$$\frac{(m_i^2 + m_j^2 + m_k^2)}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2} \Delta^{(2)} + O(g^4), \quad (3.3)$$

where $\Delta^{(2)}$ is the two-loop correction,

$$\Delta^{(2)} = -2 \sum_l [(m_l^2 / MM^\dagger) - (1/3)] T(R_l), \quad (3.4)$$

which vanishes for the universal choice [18]. Further, it was found [22] that the relation

$$h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d \ln g}, \quad (3.5)$$

among couplings is all-loop RGI. Moreover, the progress made using the spurion technique leads to all-loop relations among SSB β -functions [5, 19, 20, 21, 22], which allowed to find the all-loop RGI sum rule [17] in the Novikov-Shifman-Vainshtein-Zakharov scheme [23].

4. Search for realistic FUTs based on product gauge groups

Let us now examine the possibility of constructing realistic FUTs based on product gauge groups. Consider the gauge group $SU(N)_1 \times SU(N)_2 \times \dots \times SU(N)_k$ with n_f copies of the supersymmetric multiplet $(N, N^*, 1, \dots, 1) + (1, N, N^*, \dots, 1) + \dots + (N^*, 1, 1, \dots, N)$. The one-loop β -function coefficient in the renormalization-group equation of each $SU(N)$ gauge coupling is simply given by

$$b = \left(-\frac{11}{3} + \frac{2}{3}\right) N + n_f \left(\frac{2}{3} + \frac{1}{3}\right) \left(\frac{1}{2}\right) 2N = -3N + n_f N. \quad (4.1)$$

This means that $n_f = 3$ is a solution of the equation $b = 0$, independently of the values of N and k . Since $b = 0$ is a necessary condition for a finite field theory, the existence of three families of quarks and leptons is natural in such models. (This is true of course only if the matter content is exactly as given above. Other $SU(N)^k$ models exist with very different, and rather *ad hoc*, supermultiplet structure. They are not included in our discussion.)

Next let us examine if this class of models can meet the obvious requirements in every unified theory, namely (i) that it leads to the SM or the MSSM at low energies, and (ii) that it predicts correctly $\sin^2 \theta_W$.

Let $N = 3$ and $k = 3$, then we have the well-known example of $SU(3)_C \times SU(3)_L \times SU(3)_R$ [24, 25], with quarks transforming as

$$q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} \sim (3, 3^*, 1), \quad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} \sim (3^*, 1, 3), \quad (4.2)$$

and leptons transforming as

$$\lambda = \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix} \sim (1, 3, 3^*). \quad (4.3)$$

If we switch the first and third rows of q^c together with the first and third columns of λ , we obtain the alternative left-right model first proposed in ref. [26] in the context of superstring-inspired E_6 . The breaking down of $SU(3)^3$ to $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{Y_L+Y_R}$ is achieved with the $(3,3)$ entry of λ , and the further breaking of $SU(2)_R \times U(1)_{Y_L+Y_R}$ to $U(1)_Y$ with the $(3,1)$ entry.

Let $N = 3$ and $k = 4$, then one example is the extension to include the chiral color of ref. [27]. Here $SU(3)_C$ is split up into $SU(3)_{CL}$ and $SU(3)_{CR}$. This implies the existence of a neutral supermultiplet η transforming as (N^*, N) under these two groups. Let $\langle \eta_{11} \rangle = \langle \eta_{22} \rangle = \langle \eta_{33} \rangle$, then $SU(3)_{CL} \times SU(3)_{CR}$ breaks back down to $SU(3)_C$ as desired. However at this scale,

$$\alpha_s^{-1} = \alpha_{sL}^{-1} + \alpha_{sR}^{-1} \quad (4.4)$$

and since α_{sL} and α_{sR} are to be unified with α_L and α_R , the predicted value of α_s would be too small. Thus this is not a candidate model of unification, unless the particle content is also extended [28], in which case finiteness would be lost.

Another possibility to consider is the quartification model of ref. [29]. Here unification is possible but only in the nonsupersymmetric case. In fact, $\sin^2 \theta_W = 1/3$ instead of the canonical $3/8$, and the unification scale of this model is only 4×10^{11} GeV.

Let us now turn to the interesting $N = 4$ and $k = 3$ case [10]. The obvious choice is $SU(4)_C \times SU(4)_L \times SU(4)_R$, where $SU(4)_C$ is the Pati-Salam color gauge group [30]. In that case, the quarks and leptons should transform as

$$f = \begin{pmatrix} d & u & y & x \\ d & u & y & x \\ d & u & y & x \\ e & \nu & a & v \end{pmatrix} \sim (4, 4^*, 1), \quad f^c = \begin{pmatrix} d^c & d^c & d^c & e^c \\ u^c & u^c & u^c & \nu^c \\ y^c & y^c & y^c & a^c \\ x^c & x^c & x^c & v^c \end{pmatrix} \sim (4^*, 1, 4). \quad (4.5)$$

We see immediately that there have to be new heavy particles, i.e. the x and y quarks and the v and a leptons. In addition, we need to consider the $h \sim (1, 4, 4^*)$ supermultiplet.

The unification of quarks and leptons within $SU(4)_C$ implies that their electric charge Q should be given by

$$Q = \frac{1}{2}(B - L) + I_{3L} + I_{3R}. \quad (4.6)$$

However, the electric charges of the new heavy particles are not yet fixed. This is because $SU(4)$ contains two disjoint $SU(2)$ subgroups, one of which may be the usual $SU(2)_L$ or $SU(2)_R$, but the other is new. Therefore, another valid formula for Q is given by

$$Q = \frac{1}{2}(B - L) + I_{3L} + I_{3R} + I'_{3L} + I'_{3R}. \quad (4.7)$$

The quarks and leptons do not transform under $SU(2)'_L$ or $SU(2)'_R$, so their electric charges are not affected.

Using Eq. (4.6), the charges of f , f^c , and h are respectively

$$Q_f = \begin{pmatrix} -1/3 & 2/3 & 1/6 & 1/6 \\ -1/3 & 2/3 & 1/6 & 1/6 \\ -1/3 & 2/3 & 1/6 & 1/6 \\ -1 & 0 & -1/2 & -1/2 \end{pmatrix}, \quad (4.8)$$

$$Q_{f^c} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 1 \\ -2/3 & -2/3 & -2/3 & 0 \\ -1/6 & -1/6 & -1/6 & 1/2 \\ -1/6 & -1/6 & -1/6 & 1/2 \end{pmatrix}, \quad (4.9)$$

$$Q_h = \begin{pmatrix} 0 & 1 & 1/2 & 1/2 \\ -1 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \end{pmatrix}. \quad (4.10)$$

Using Eq. (4.7), they are instead

$$Q_f = \begin{pmatrix} -1/3 & 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 & 2/3 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad (4.11)$$

$$Q_{f^c} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & -1 \\ -2/3 & -2/3 & -2/3 & 0 \\ 1/3 & 1/3 & 1/3 & -1 \\ -2/3 & -2/3 & -2/3 & 0 \end{pmatrix}, \quad (4.12)$$

$$Q_h = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \quad (4.13)$$

The two different charge assignments result in two different values of

$$\sin^2 \theta_W = \frac{\sum I_{3L}^2}{\sum Q^2} \quad (4.14)$$

at the unification scale. Whereas it is equal to $3/8$ as usual in the former, it becomes $3/14$ in the latter, which is not realistic. Therefore we will discuss further only the case with the charge assignments of Eqs. (4.8–4.10).

Since we do not admit any other matter supermultiplets, the symmetry breaking of $SU(4)_C \times SU(4)_L \times SU(4)_R$ must be achieved with the vacuum expectation values of the neutral scalar components of f , f^c , and h . The best we can do is to let all the (3,3), (3,4), (4,3), and (4,4) entries of h acquire vacuum expectation values, but then the $SU(4)^3$ symmetry is only broken down to $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$. The extra unwanted $U(1)$ is necessarily present because in the decomposition of $SU(4)_L$ and $SU(4)_R$ to their $SU(2) \times SU(2) \times U(1)$ subgroups, the diagonal subgroup $U(1)_{L+R}$ cannot be broken by the representation (1,4,4*). This problem persists even after the breaking of $SU(4)_C \times SU(2)_R$ by the (2,4) entry of f^c to $SU(3)_C \times U(1)_Y$.

Since the unbroken $U(1)$ couples to all particles, including the known quarks and leptons, this model cannot be viable phenomenologically. We are thus forced to conclude that $SU(4)_C \times SU(4)_L \times SU(4)_R$ with only the matter content of f , f^c , and h is not a suitable candidate for a finite theory of all particles.

There is another important constraint for a realistic $SU(N)^k$ theory of quarks and leptons, i.e. the proper masses must be obtained. Excluding naturally nonrenormalizable terms in the superpotential, then only bilinear and trilinear terms are allowed. For the matter content assumed here, it would be zero unless $N = 3$ or $k = 3$. (We exclude $N = 2$ or $k = 2$ for obvious reasons.) If $N = 3$, then we have an invariant from the product of three (3,3*) supermultiplets. If $k = 3$, then the invariant $(N, N^*, 1)(1, N, N^*)(N^*, 1, N)$ may be formed. Therefore, this discussion leads us naturally to the case $SU(3)^3$.

5. An all-loop $SU(3)^3$ FUT

Here we will discuss in some detail the supersymmetric $SU(3)^3$ FUT with three families. In general a supersymmetric E_6 model in four dimensions is easily obtained in compactifications of a ten-dimensional E_8 , appearing in the heterotic string, over Calabi-Yau spaces [31]. Even more interesting is the possibility to obtain softly broken supersymmetric E_6 type models via coset space dimensional reduction [32, 33] in compactifications using non-symmetric coset spaces [34]. Subsequently the $SU(3)^3$ can emerge using the Wilson fluxes [31, 35] in a straightforward way. What is less obvious to obtain is the spontaneous symmetry breaking of $SU(3)^3$ down to the MSSM, however it has been done already some time ago [36]. It requires introducing eight superfield of the type (λ, q, q^c) and five corresponding mirror superfields $(\bar{\lambda}, \bar{q}, \bar{q}^c)$. The details of this construction are given in ref. [36]. Therefore what remains as an open question is how to obtain the complete and detailed chain of breakings of the ten-dimensional E_8 down to the four-dimensional MSSM, but this is deeply related to the most fundamental problem of string theory, and will not be addressed further here. For our purposes, following [36], we consider a supersymmetric $SU(3)^3$ model with three families holding between the Planck M_P and the unification M_{GUT} scales, which breaks spontaneously down to the MSSM at M_{GUT} .

In order for all the gauge couplings to be equal at M_{GUT} , as is suggested by the LEP results [37], the cyclic symmetry Z_3 must be imposed, i.e.

$$q \rightarrow \lambda \rightarrow q^c \rightarrow q, \quad (5.1)$$

where q and q^c are given in Eq. (4.2) and λ in Eq. (4.3). Then, according to the discussion in section 3, the first of the finiteness conditions (2.5) for one-loop finiteness, namely the vanishing of the gauge β -function is satisfied.

Next let us consider the second condition, i.e. the vanishing of the anomalous dimensions of all superfields. To do that first we have to write down the superpotential. If there is just one family, then there are only two trilinear invariants, which can be constructed respecting the symmetries of the theory, and therefore can be used in the superpotential as follows

$$f \text{Tr}(\lambda q^c q) + \frac{1}{6} f' \epsilon_{ijk} \epsilon_{abc} (\lambda_{ia} \lambda_{jb} \lambda_{kc} + q_{ia}^c q_{jb}^c q_{kc}^c + q_{ia} q_{jb} q_{kc}). \quad (5.2)$$

In this case, the condition for vanishing anomalous dimension of each superfield is given by [1, 2, 3, 4, 5]

$$\frac{1}{2}(3|f|^2 + 2|f'|^2) = 2 \left(\frac{4}{3} g^2 \right). \quad (5.3)$$

Quark and leptons obtain masses when the scalar parts of the superfields (\tilde{N}, \tilde{N}^c) obtain vacuum expectation values (vevs),

$$m_d = f \langle \tilde{N} \rangle, \quad m_u = f \langle \tilde{N}^c \rangle, \quad m_e = f' \langle \tilde{N} \rangle, \quad m_\nu = f' \langle \tilde{N}^c \rangle. \quad (5.4)$$

With three families, the most general superpotential contains 11 f couplings, and 10 f' couplings, subject to 9 conditions, due to the vanishing of the anomalous dimensions of each superfield. The conditions are the following

$$\sum_{j,k} f_{ijk} (f_{ljk})^* + \frac{2}{3} \sum_{j,k} f'_{ijk} (f'_{ljk})^* = \frac{16}{9} g^2 \delta_{il}, \quad (5.5)$$

where

$$f_{ijk} = f_{jki} = f_{kij}, \quad (5.6)$$

$$f'_{ijk} = f'_{jki} = f'_{kij} = f'_{ikj} = f'_{kji} = f'_{jik}. \quad (5.7)$$

Quarks and leptons receive masses when the scalar part of the superfields $\tilde{N}_{1,2,3}$ and $\tilde{N}_{1,2,3}^c$ obtain vevs as follows

$$(\mathcal{M}_d)_{ij} = \sum_k f_{kij} \langle \tilde{N}_k \rangle, \quad (\mathcal{M}_u)_{ij} = \sum_k f_{kij} \langle \tilde{N}_k^c \rangle, \quad (5.8)$$

$$(\mathcal{M}_e)_{ij} = \sum_k f'_{kij} \langle \tilde{N}_k \rangle, \quad (\mathcal{M}_\nu)_{ij} = \sum_k f'_{kij} \langle \tilde{N}_k^c \rangle. \quad (5.9)$$

Since we want to have, among other conditions, gauge coupling unification, we will assume that the particle content of our finite $SU(3)^3$ model below M_U is that of the MSSM with three fermion families, but only two Higgs doublets. Therefore we have to choose the linear combinations $\tilde{N}^c = \sum_i a_i \tilde{N}_i^c$ and $\tilde{N} = \sum_i b_i \tilde{N}_i$ to play the role of the two Higgs doublets, which will be responsible for the electroweak symmetry breaking. This can be done by choosing appropriately the masses in the superpotential [38], since they are not constrained by the finiteness conditions. Moreover, we choose that the two Higgs doublets

are predominately coupled to the third generation. Then these two Higgs doublets couple to the three families differently, thus providing the freedom to understand their different masses and mixings.

Assuming for our purposes here that all f' couplings vanish¹ an isolated solution Eq. (5.5) is

$$f^2 = f_{111}^2 = f_{222}^2 = f_{333}^2 = \frac{16}{9}g^2. \quad (5.10)$$

Hence we start at M_{GUT} with different Yukawa couplings for all the quarks

$$f_t = fa_3, \quad f_c = fa_2, \quad f_u = fa_1, \quad (5.11)$$

$$f_b = fb_3, \quad f_s = fb_2, \quad f_d = fb_1, \quad (5.12)$$

which is similar to the MSSM except that f is fixed by finiteness at M_{GUT} , and $a_3 \simeq 1$, $b_3 \simeq 1$, by construction, and therefore we have that $f_t \simeq f_b \simeq f$ at M_{GUT} . As for the lepton masses, because all f' couplings have been fixed to be zero at this order, in principle they are expected to appear radiatively induced by the scalar lepton masses appearing in the SSB sector of the theory. Unfortunately though, due to the finiteness conditions (3.2) they cannot appear radiatively and remain as a problem for further study. On the other it should be stressed that we can certainly let f' be non-vanishing in Eq. (5.5) and thus introduce lepton masses in the model. Then the real price to be paid is basically aesthetic since the model in turn becomes finite only up to two-loops since the corresponding solution of Eq. (5.5) is not an isolated one any more. However, given that the analysis we do in the next section takes into account RGEs up to two-loops, there is no practical cost in introducing non-zero f' . We include this possibility in our analysis in section 6.

Although we present the results of a more complete analysis in the next section, we find instructive to describe here the situation concerning the top quark mass prediction at one-loop level ignoring the SSB sector. In this approximate analysis, we run the MSSM renormalization group equations at one-loop, using our boundary condition $f^2 = (16/9)g^2$ at the M_{GUT} scale as follows

$$8\pi^2(dg_3^2/dt) = -3g_3^4, \quad (5.13)$$

$$8\pi^2(dg_2^2/dt) = g_2^4, \quad (5.14)$$

$$8\pi^2(dg_1^2/dt) = \frac{33}{5}g_1^4, \quad (5.15)$$

$$8\pi^2(df_t^2/dt) = f_t^2 \left(6f_t^2 + f_b^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2 \right), \quad (5.16)$$

$$8\pi^2(df_b^2/dt) = f_b^2 \left(6f_b^2 + f_t^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2 \right). \quad (5.17)$$

The g_i^2 s are easily solved as functions of $t = \ln(M_{GUT}/M)$:

$$\alpha_3(M)^{-1} = \alpha_3(M_{GUT})^{-1} - (3/2\pi) \ln(M_{GUT}/M), \quad (5.18)$$

¹In supersymmetric theories this can always be done due to the non-renormalization theorem [39], which guarantees that these terms will not appear radiatively. In general this is not the case in the presence of supersymmetry breaking terms, however finiteness imposes tight conditions in this respect too.

$$\alpha_2(M)^{-1} = \alpha_2(M_{GUT})^{-1} + (1/2\pi) \ln(M_{GUT}/M), \quad (5.19)$$

$$\alpha_1(M)^{-1} = \alpha_1(M_{GUT})^{-1} + (33/10\pi) \ln(M_{GUT}/M), \quad (5.20)$$

where $\alpha_i = g_i^2/4\pi$. Using the MSSM boundary conditions from the unification of the gauge couplings at one-loop and the constraints of the present model we have

$$\alpha_i(M_{GUT}) = 0.0413, \quad (5.21)$$

$$\alpha_t(M_{GUT}) = \alpha_b(M_{GUT}) = (16/9)\alpha_i(M_{GUT}). \quad (5.22)$$

Then we integrate the two differential equations (5.16) and (5.17), from $t = \ln(M_{GUT}/M_{EW})$ to $t = 0$, to determine f_t and f_b at the electroweak scale M_{EW} . Then $m_t = f_t v_u$ and $m_b = f_b v_d$, with v_u and v_d satisfying the condition $v_u^2 + v_d^2 = v^2$, $v = 174.3$ GeV. Thus given m_b , we can obtain m_t .

6. Predictions and Conclusions

The gauge symmetry $SU(3)^3$ is spontaneously broken down to the MSSM at M_{GUT} , and the finiteness conditions do not restrict the renormalization properties at low energies. Therefore, below M_{GUT} all couplings and masses of the theory run according to the RGEs of the MSSM. The remnants of the all-loop FUT $SU(3)^3$ are the boundary conditions on the gauge and Yukawa couplings (5.10), the $h = -MC$ relation, and the soft scalar-mass sum rule (3.3) at M_{GUT} , which, when applied to the present model, takes the form

$$m_{H_u}^2 + m_{t^c}^2 + m_{\bar{q}}^2 = M^2 \quad (6.1)$$

$$m_{H_d}^2 + m_{b^c}^2 + m_{\bar{q}}^2 = M^2. \quad (6.2)$$

Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones imposing the corresponding boundary conditions. We further assume a unique supersymmetry breaking scale M_s (defined as the average of the mass of the stops) and therefore below that scale the effective theory is just the SM.

We consider two versions of the model:

I) The all-loop finite one in which f' vanishes and Eq. (5.10) holds.

II) A two-loop finite version, in which we keep f' non-vanishing in Eq. (5.5), and we use it to introduce the lepton masses.

The predictions for the top quark mass m_t are ~ 183 GeV for $\mu < 0$ in model **I**, whereas for model **II** it is 176 - 179 GeV for $\mu < 0$, and 170 - 173 GeV for $\mu > 0$. Recall that the bottom quark mass m_b is an input in **FUT I** and m_τ in **FUT II**.

Comparing these predictions with the most recent experimental value $m_t^{exp} = (178.0 \pm 4.3)$ GeV [40], and recalling that the theoretical values for m_t may suffer from a correction of $\sim 4\%$ [5], we see that they are consistent with the experimental data.

In the SSB sector, besides the constraints imposed by finiteness we further require

- 1) successful radiative electroweak symmetry breaking, and
- 2) $m_{\tilde{\tau}, \tilde{b}, \tilde{t}}^2 > 0$.

As an additional constraint, we take into account the $BR(b \rightarrow s\gamma)$ [41]. We do not take into account, though, constraints coming from the muon anomalous magnetic moment (g-2) in this work, which would exclude a small region of the parameter space.

Our numerical analysis shows the following results for the two models: In the case of **FUT I** it is possible to find regions of parameter space which comply with all the above requirements both for the case where we have universal boundary conditions ($m_i^2 = m_j^2 = m_k^2 = M^2/3$), and for the case where we apply the sum rule Eq.(3.3). In the case of universal boundary conditions and $\mu < 0$, $m_t \sim 183$ GeV, the Higgs mass is $\sim 131 - 132$ GeV, $\tan \beta \sim 50 - 51$, and the spectrum is rather heavy, the allowed region of parameter space starting with an LSP which is a neutralino $m_{\chi^0} \sim 825$ GeV for a value of $M \sim 1800$ GeV. In the case the sum rule is applied we have one more free parameter, which is $m_{\tilde{q}^c} = m_{\tilde{q}}$ at the GUT scale. In this case we obtain a $\tan \beta \sim 47 - 54$, and the Higgs mass is $\sim 130 - 132$ GeV. The main difference between the universal boundary conditions and the sum rule comes in the sparticle spectrum, which can now start with an LSP at $m_{\chi^0} \sim 450$ GeV, for a boundary condition of $M \sim 1800$ GeV. In the case that $\mu > 0$ we do not find solutions which satisfy all the above requirements.

In the second version of the model **FUT II**, we have the following boundary conditions for the Yukawa couplings

$$f^2 = r(16/9)g^2, \quad (6.3)$$

$$f'^2 = (1-r)(8/3)g^2. \quad (6.4)$$

In this case, we do not have an all-loop finite model, since the solution is a parametric one, but it is the price we pay to give masses to the leptons. As for the boundary conditions of the soft scalars, we only have the universal case. This is because, applying the sum rule (3.3) to the superpotential with $f' \neq 0$ implies that $m_q^2 = m_{q^c}^2 = m_{H^{u,d}}^2 = M^2/3$, which is again the universal boundary condition. For the numerical analysis we fix the $m_{\tilde{\tau}}$ mass to obtain m_t and m_b . Taking $\mu < 0$, and for the experimentally allowed value of $m_b(m_b) = 4.1 - 4.4$ GeV [42], the value of m_t goes from $\sim 176 - 179$ GeV. In this case $\tan \beta \sim 48 - 53$, and $m_H \sim 122 - 129$ GeV, with a charged LSP $m_{\tilde{\tau}} \sim 400 - 1000$ GeV, depending directly on the value of M , which varies from $\sim 1200 - 2200$ GeV in this case.

Now for $\mu > 0$, the value of m_t compatible with the experimentally allowed value of m_b goes from $\sim 170 - 173$ GeV, clearly the preferred value being the latter. For this range of values of m_t we obtain $\tan \beta \sim 58 - 62$, and $m_H \sim 120 - 125$ GeV, also with a charged LSP $m_{\tilde{\tau}} \sim 300 - 600$ GeV, again depending directly on the value of M , which varies from $\sim 1300 - 2000$ GeV.

We could go further and consider another version of the $SU(3)^3$ model. For instance, if we impose global $SU(3)$ as a family symmetry [11, 43], then there is only one Yukawa coupling in the superpotential, which leads to the following unique relation among Yukawa and gauge couplings

$$f^2 = \frac{8}{9}g^2. \quad (6.5)$$

However both \mathcal{M}_u and \mathcal{M}_d in Eq. (5.8) must now be antisymmetric in family space, resulting in one zero and two equal mass eigenvalues for each, which is not a realistic case.

Note moreover, that the terms proportional to f' in the superpotential Eq. (5.2) are not allowed to appear in the cases of refs. [10, 11] unless $N = 3$, and therefore they share the problem of the **FUT I** model, where we have chosen $f' = 0$.

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